

Fokker-Planck Equation on Fractal Curves

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Abstract

A Fokker-Planck equation on fractal curves is obtained, starting from Chapmann-Kolmogorov equation on fractal curves. This is done using the recently developed calculus on fractals, which allows one to write differential equations on fractal curves. As an important special case, the diffusion and drift coefficients are obtained, for a suitable transition probability to get the diffusion equation on fractal curves. This equation is of first order in time, and, in space variable it involves derivatives of order α , α being the dimension of the curve. The solution of this equation with localized initial condition shows deviation from ordinary diffusion behaviour due to underlying fractal space in which diffusion is taking place. An exact solution of this equation manifests a subdiffusive behaviour. The dimension of the fractal path can be estimated from the distribution function.

Fractal curves and paths are encountered frequently in Physics [1, 2]. Several geometries like that of polymer chains, percolating clusters, Brownian and Fractional Brownian trajectories and many more have been identified as fractals, more precisely as fractal curves [1]. Transport on these structures reveal remarkable properties [3, 4, 5, 6, 7, 8, 9]. In particular, anomalous diffusion on fractals is a topic of immense current interest [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. There have been several investigations [20, 21, 22, 23], both analytical as well as numerical which shed light on the various facets of the problem. These investigations use the Fractional operators. The Fractional Derivatives are often used to explore the characteristic features of anomalous diffusion by setting up fractional kinetic equations. Fractional derivatives are non local operators and hence not always suitable to handle local scaling behaviour. Analysis on fractals is another remarkable development which has been extensively used for the treatment of diffusion, heat conduction, waves etc on fractals. But, in this approach the operators are constructed using the self similarity of fractal sets, and these are restricted to such sets which are post critically finite. However, as is well known, an ordinary diffusion equation is inadequate to describe anomalous transport, and an exact equation based on an appropriately developed calculus is still desired.

Ordinary calculus does not equip us to handle problems such as anomalous diffusion, dynamics on fractals, fields of fractally distributed sources etc by setting up and solving ordinary differential equations. Several authors have recognized the need to use fractional derivatives and integrals to explore the characteristic features of fractal walks, anomalous diffusion, transport etc. by setting up fractional kinetic equations, master equations and so on.

Further measure-theoretical approaches are used which include defining derivative as inverse of the integral with respect to a measure and defining other operators using derivatives. Even though measure theoretical approach is elegant, Riemann integration like procedures have their own place. They are more transparent, constructive and advantageous from algorithmic point of view.

Fractal curves lack the smoothness properties required by ordinary calculus. Recently, [24] a new calculus on fractal curves, such as the von Koch curve, was formulated. In this calculus, a Riemann-like integral along a fractal curve F is defined. This integral is called the F^α -integral, where α is the dimension of the curve. A derivative along the fractal curve, called F^α -derivative, is also defined. These operators are different from the fractional operators used in [10, 11, 12], in the sense that they are local and a newly defined measure like quantity called the mass function (which is algorithmic in nature) is used to define these. The order of these operators is exactly equal to the dimension of underlying space, and thus they reduce to ordinary integral and derivative operators when the space is \mathbf{R} and not a fractal.

Several aspects of this calculus retain much of the simplicity of ordinary calculus. In fact a conjugacy between this calculus and ordinary calculus on the real line is established. This new calculus which is simple, direct and algorithmic can be applied to various physical processes.

Here, using the framework of this calculus, we develop a first-principles approach to the Fokker-Planck equation on fractal curves. A particular choice of transition probability then leads to a new form of diffusion equation. This equation is of the first order in time and involves application of the F^α -derivative with respect to spatial variable twice. An exact solution of this equation shows a subdiffusive behaviour.

We begin by fixing our notation as in [24]. We consider a (fractal) curve $F \subset \mathbf{R}^n$ which is continuously parametrizable i.e there exists a function $\mathbf{w} : [a_0, b_0] \rightarrow F \subset \mathbf{R}^n$ which is continuous. We also assume \mathbf{w} to be invertible. A subdivision $P_{[a,b]}$ of interval $[a, b]$, $a < b$, is a finite set of points $\{a = u_0 < u_1, \dots < u_n = b\}$. For $a_0 \leq a < b \leq b_0$ and appropriate α to be chosen, let

$$\gamma^\alpha(F, a, b) = \lim_{\delta \rightarrow 0} \inf_{\{P_{[a,b]} : |P| \leq \delta\}} \sum_{i=0}^{n-1} \frac{|\mathbf{w}(u_{i+1}) - \mathbf{w}(u_i)|^\alpha}{\Gamma(\alpha + 1)} \quad (1)$$

where $|\cdot|$ denotes the Euclidean norm on \mathbf{R}^n . and $|P| = \max\{u_{i+1} - u_i; i = 0, \dots, n-1\}$ A new dimension, the γ -dimension of F , which will be denoted by $\dim_\gamma(F)$, is given by

$$\dim_\gamma(F) = \inf\{\alpha : \gamma^\alpha(F, a, b) = 0\} = \sup\{\alpha : \gamma^\alpha(F, a, b) = \infty\}$$

Hereafter, α will be assumed to be equal to $\dim_\gamma(F)$ (thus, $\alpha \geq 1$).

The rise (staircase) function $S_F^\alpha : [a_0, b_0] \rightarrow \mathbf{R}$ of order α for a set F , is defined as

$$S_F^\alpha(u) = \begin{cases} \gamma^\alpha(F, p_0, u) & u \geq p_0 \\ -\gamma^\alpha(F, u, p_0) & u < p_0 \end{cases} \quad (2)$$

where $a_0 \leq p_0 \leq b_0$ is arbitrary but fixed, and $u \in [a_0, b_0]$. It is a monotonic function. We denote a point on the fractal curve F by $\theta = \mathbf{w}(u)$, and define

$$J(\theta) = S_F^\alpha(\mathbf{w}^{-1}(\theta)), \quad \theta \in F$$

Hereafter we consider only those curves for which S_F^α is *finite* and *invertible* on $[a, b]$. The F^α -derivative of a bounded function $f : F \rightarrow \mathbf{R}$ at $\theta \in F$ is defined as

$$(D_F^\alpha f)(\theta) = F\text{-}\lim_{\theta' \rightarrow \theta} \frac{f(\theta') - f(\theta)}{J(\theta') - J(\theta)} \quad (3)$$

where the F -lim denotes limit along points of F , if the limit exists.

Let $C(a, b)$ denote the segment $\{\mathbf{w}(u) : u \in [a, b]\}$ of F . A Riemann-like integral on F , called F^α -integral, is also defined [24]. It is denoted by $\int_{C(a, b)} f(\theta) d_F^\alpha \theta$. The above mentioned F^α -derivative and F^α -integral are related to each other through the Fundamental Theorems of Calculus as “inverses” of each other [24].

The notion of conjugacy of calculus on fractals and ordinary calculus on the real line is very useful. Let ϕ denote the map (conjugacy) from the class of bounded functions on F to the class of bounded functions on the interval $[S_F^\alpha(a_0), S_F^\alpha(b_0)]$ defined by $\phi[f](S_F^\alpha(u)) = f(\mathbf{w}(u))$. Then it follows [24]

$$\int_{S_F^\alpha(a)}^{S_F^\alpha(b)} g(x) dx = \int_{C(a, b)} f(\theta) d_F^\alpha \theta \quad (4)$$

where $g = \phi[f]$. Moreover ϕ relates derivatives D_F^α with the ordinary derivative D , thus $(D_F^\alpha f)\mathbf{w}(u) = (Dg)(u)$.

The Taylor series is given by

$$h(\theta) = \sum_{n=0}^{\infty} \frac{(J(\theta) - J(\theta'))^n}{n!} (D_F^\alpha)^n h(\theta') \quad (5)$$

provided the bounded function h is F^α -differentiable any number of times on $C(a, b)$. It is also possible to write a Taylor series with remainder.

Let $\int_{C(a, b)} V(\theta, t) d_F^\alpha \theta$ be the probability that a particle constrained to move on F is found in the segment $C(a, b)$, or in other words, $V(\theta, t)$ denotes the ‘fractal’ probability density that the particle is found at θ at time t . Let the probability density for transition from a point θ' at time t to θ at time $t + \tau$, be denoted by $P(\theta, t + \tau | \theta', t)$. A formalism to analyse similar situation in ordinary space is developed in [25], we intend to modify the same for the case of fractal curves.

The Chapman-Kolmogorov equation on fractal curve F can be written in the form

$$V(\theta, t + \tau) = \int_{C(a,b)} P(\theta, t + \tau | \theta', t) V(\theta', t) d_F^\alpha \theta' \quad (6)$$

where $\theta, \theta' \in F$. Let $\Delta \equiv \Delta(\theta, \theta') = J(\theta) - J(\theta')$. The integrand in equation (6) is:

$$\begin{aligned} P(\theta, t + \tau | \theta', t) V(\theta', t) &= P(J^{-1}(J(\theta) - \Delta + \Delta), \\ &\quad t + \tau | J^{-1}(J(\theta) - \Delta), t) \\ &\quad \times V(J^{-1}(J(\theta) - \Delta), t) \end{aligned}$$

The Taylor expansion of this integrand in (6) then leads to

$$\begin{aligned} &V(\theta, t + \tau) - V(\theta, t) \\ &= \int \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \Delta^n (D_F^\alpha |_\theta)^n \{ P(J^{-1}(J(\theta) + \Delta), \\ &\quad t + \tau | \theta, t) V(\theta, t) \} d_F^\alpha \theta' \end{aligned} \quad (7)$$

While there are other ways to write the Fokker-Planck equation we turn to the use of conjugacy which reduces the problem to the ordinary case and the meaning of moments will be transparent. We use the following explicit notation for the conjugacy map ϕ :

$$(\phi_\theta H)(J(\theta), \theta') = H(\theta, \theta')$$

$$(\phi_{\theta'} H)(\theta, J(\theta')) = H(\theta, \theta')$$

while applying to a function $H(\theta, \theta')$ of two arguments θ and θ' .

Let $y = J(\theta), y' = J(\theta')$ and assume $S_F^\alpha(a_0) \leq y, y' \leq S_F^\alpha(b_0)$. Further let us denote

$$\begin{aligned} \tilde{\Delta}(\theta, y') &= \phi_{\theta'} \Delta(\theta, \theta') \\ \Delta'(y, y') &= \phi_\theta \tilde{\Delta} = \phi_\theta \circ \phi_{\theta'} \Delta(\theta, \theta') \end{aligned}$$

Then,

$$\tilde{\Delta}(\theta, y') = J(\theta) - y' \text{ and } \Delta'(y, y') = y - y'$$

where $S_F^\alpha(a_0) \leq y, y' \leq S_F^\alpha(b_0)$.

Now we define $V' = \phi_\theta(V)$ and $P' = \phi_{\theta'} \circ \phi_\theta P$. Using the conjugacy of integrals from equation (4), equation (7) becomes:

$$\begin{aligned} V'(y, t + \tau) - V'(y, t) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{S_F^\alpha(a_0)}^{S_F^\alpha(b_0)} (\Delta')^n \\ &\quad \left(\frac{\partial}{\partial y} \right)^n \{ P'(y + \Delta', t + \tau | y, t) V'(y, t) \} dy' \end{aligned} \quad (8)$$

Integrating over Δ' , we see that $dy' = -d\Delta'$ Hence,

$$\begin{aligned} V'(y, t + \tau) - V'(y, t) &= - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_{y-S_F^\alpha(a_0)}^{y-S_F^\alpha(b_0)} (\Delta')^n \\ &\quad \left(\frac{\partial}{\partial y} \right)^n \{ P'(y + \Delta', t + \tau | y, t) - V'(y, t) \} d\Delta' \end{aligned} \quad (9)$$

The transitional moments are given by

$$\tilde{M}_n(y, t, \tau) = \phi_{\theta'} M_n(\theta', t, \tau) \quad (10)$$

$$= \int_{y-S_F^\alpha(b_0)}^{y-S_F^\alpha(a_0)} (\Delta')^n P'(y + \Delta', t + \tau | y, t) d\Delta' \quad (11)$$

Hence

$$M_n(\theta', t, \tau) = \int_{C(a_0, b_0)} (J(\theta) - J(\theta'))^n P(\theta, t + \tau | \theta', t) d_F^\alpha \theta \quad (12)$$

Substituting (10) in equation (9), and applying conjugacy we get

$$V(\theta, t + \tau) - V(\theta, t) = \sum_{n=1}^{\infty} (-D_F^\alpha)^n \left\{ \frac{M_n(\theta, t, \tau)}{n!} V(\theta, t) \right\}$$

Now we assume that the moments \tilde{M}_n and M_n can be expanded in a Taylor series.

$$\frac{\tilde{M}_n(y, t, \tau)}{n!} = (\tilde{A})^{(n)}(y, t) \tau + O(\tau^2) \quad (13)$$

and

$$\frac{M_n(\theta, t, \tau)}{n!} = A^{(n)}(\theta, t) \tau + O(\tau^2) \quad (14)$$

The term of order τ^0 vanishes because for $\tau = 0$ the transition probability is $P'(y, t | y', t) = \delta(y, y')$ which leads to vanishing moments. By taking into account only the linear terms in τ we have the Kramers-Moyal expansion

$$\frac{\partial}{\partial t} V'(y, t) = \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial y} \right)^n \{ (\tilde{A})^{(n)}(y, t) V'(y, t) \} \quad (15)$$

i.e.

$$\frac{\partial}{\partial t} V(\theta, t) = \sum_{n=1}^{\infty} (-D_F^\alpha)^n \{ A^{(n)}(\theta, t) V(\theta, t) \} \quad (16)$$

From the above we obtain the Fokker-Planck equation if the expansion in equation (16) stops after second term,

Thus

$$\frac{\partial V(\theta, t)}{\partial t} = -D_F^\alpha \{ A^{(1)}(\theta, t) V(\theta, t) \} + \quad (17)$$

$$(-D_F^\alpha)^2 \{ A^{(2)}(\theta, t) V(\theta, t) \} \quad (18)$$

where $A^{(1)}$ is the (fractal) drift coefficient and $A^{(2)}$ is the (fractal) diffusion coefficient.

Now we consider the special case with Gaussian transition probability

$$P(\theta, t + \tau | \theta', t) = \frac{1}{\sqrt{\pi\tau}} \exp\left\{-\frac{(J(\theta) - J(\theta'))^2}{\tau}\right\}$$

Then from equation (12)

$$M_n(\theta', t, \tau) = \frac{1}{\sqrt{\pi\tau}} \int_{C(a_0, b_0)} (J(\theta) - J(\theta'))^n \exp\left\{-\frac{(J(\theta) - J(\theta'))^2}{\tau}\right\} d_F^\alpha \theta$$

The conjugate equation for moments gives

$$\tilde{M}_n(y', t, \tau) = \frac{1}{\sqrt{\pi\tau}} \int_{S_F^\alpha(a_0)}^{S_F^\alpha(b_0)} (y - y')^n \exp\left\{-\frac{(y - y')^2}{\tau}\right\} dy$$

for $S_F^\alpha(a_0) \ll y'$ and $S_F^\alpha(b_0) \gg y'$ we may replace the limits of the above integral by $-\infty$ and $+\infty$ respectively to get $\tilde{M}_1 = 0$ and $\tilde{M}_2 = \tau/2$ For $n = 1$ and $n = 2$ using equation (13) we see that $\tilde{A}^{(1)} = 0$, $\tilde{A}^{(2)} = \frac{1}{4}$ hence the first term on the RHS of equation (17) vanishes and $A^{(2)}$ is a constant. We can then write equation (17) as

$$\frac{\partial}{\partial t} V(\theta, t) = A(D_F^\alpha)^2 V(\theta, t) \quad (19)$$

This is a new diffusion equation on the fractal curve F with “fractal” diffusion coefficient A .

The (19) is conjugate to

$$\frac{\partial V'(y, t)}{\partial t} = A \frac{\partial^2}{\partial y^2} V'(y, t) \text{ where } V' = \phi[V]$$

Given the initial condition $V'(y, 0) = \delta(y)$, the solution of the above equation is :

$$V'(y, t) = \frac{1}{\sqrt{2\pi A t}} \exp\left(-\frac{y^2}{2 A t}\right)$$

applying ϕ^{-1}

$$V(\theta = \mathbf{w}(u), t) = \frac{1}{\sqrt{2\pi A t}} \exp\left(-\frac{(J(\theta) - S_F^\alpha(u))^2}{2 A t}\right)$$

which gives the probability density V at a location $\theta = \mathbf{w}(u)$ at time t . This is an exact solution of the diffusion equation (19) on a fractal curve F . The corresponding plots for the above distribution when the curve is a von-koch

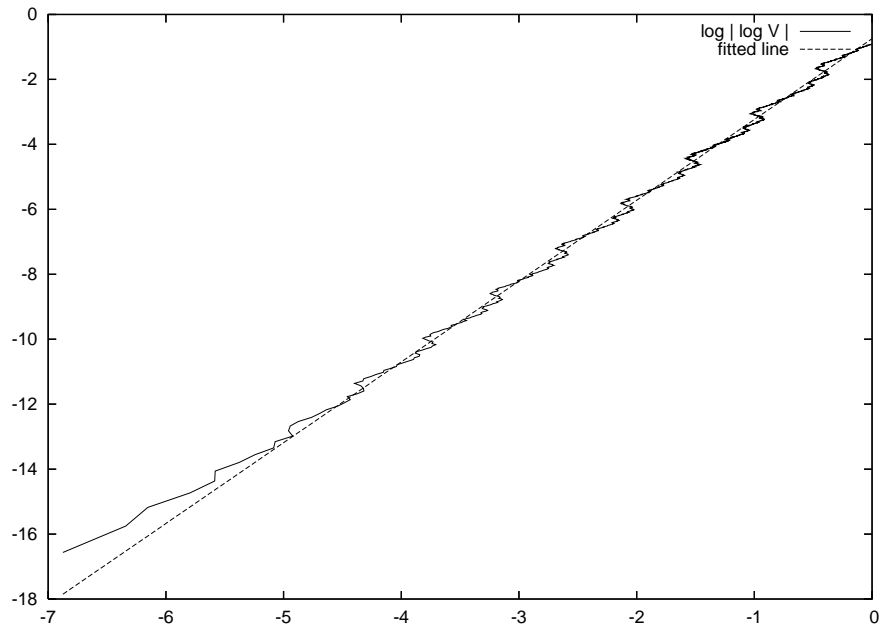


Figure 1: Plot of $(\log |\log V(\theta, t)|)$ against $(\log |\theta|)$, for a fixed t , and a straight line fit for it. The slope of the line is 2.4885, which is reasonably close to 2α , where $\alpha = \log(4)/\log(3) = 1.26$ is the dimension of the von Koch curve.

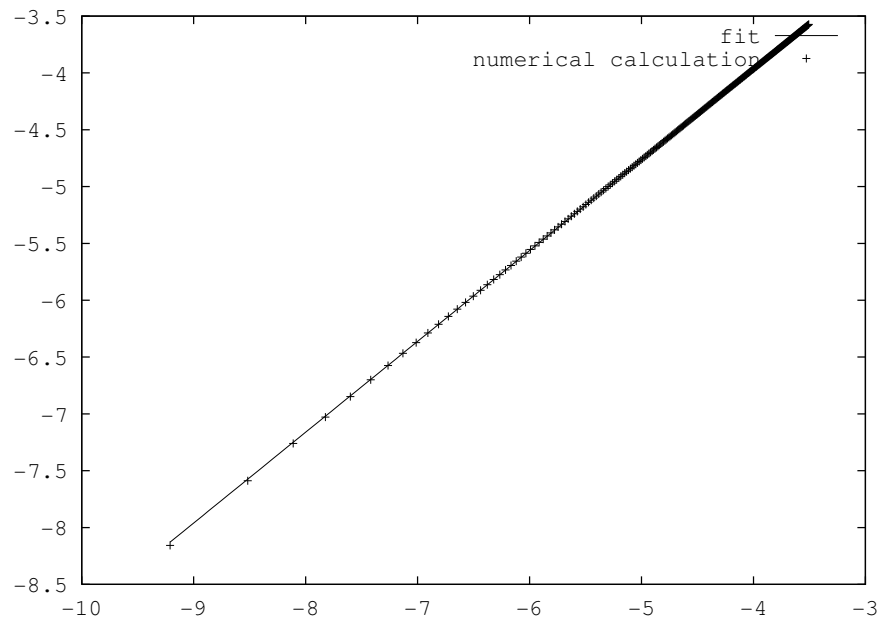


Figure 2: Plot of $\log(t)$ vs $\log(\langle L^2 \rangle)$. The slope of the line is 0.799 which is reasonably close to $1/\alpha$, $\alpha = \log(4)/\log(3) = 1.26$, being the dimension of the von-koch curve.

curve in \mathbf{R}^2 are shown in the figure 1, from which the implicit subdiffusive behaviour is clear.

In figure 2 the subdiffusive behaviour of motion on a fractal curve F is shown. F is the von-koch curve with dimension $\log(4)/\log(3) = 1.26$. The relation between Euclidean distance $L(\theta)$ and time 't' is given by

$$\int_{C(a,b)} L(\theta)^2 P(\theta, t) d_F^\alpha \theta \sim t^\mu$$

where the exponent μ decides the nature of diffusion. We find that in the above calculation, $\mu \sim 1/\alpha$, where $\alpha = 1.26$, more appropriately

$$\langle L^2 \rangle \sim t^{0.802} \sim t^{1/\alpha} \quad (20)$$

and hence $\mu < 1$ indicates the subdiffusive behaviour as a result of underlying fractal nature of space on which the particle moves.

We conclude that the underlying fractal nature of space gives rise to subdiffusive behaviour of the diffusing entity. It is rather a deviation from gaussian distribution, which would have been exactly Gaussian, had the underlying space been ordinary and not fractal in nature. Also we see that the dimension of the fractal curve can be estimated from the plots of the distribution function.

Summarizing: We have proposed a Fokker-Planck equation on fractal curves. An exact solution of the diffusion equation on such curves is seen to have a subdiffusive character. The dimension of the curve can be estimated from the diffusion function.

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